

AF LABELED GRAPH C^* -ALGEBRAS

JA A JEONG[†], EUN JI KANG[†], AND SUN HO KIM[‡]

ABSTRACT. It is known that a graph C^* -algebra $C^*(E)$ is AF if and only if the graph E has no loops. In this paper we consider the question of when a labeled graph C^* -algebra $C^*(E, \mathcal{L}, \mathcal{B})$ is AF. A notion of loop in a labeled space is defined when the accommodating set \mathcal{B} is closed under relative complements, and it is proved that if a labeled graph C^* -algebra is AF, the labeled space has no loops. A sufficient condition for a labeled graph C^* -algebra to be AF is given and various examples are discussed.

1. INTRODUCTION

For graph C^* -algebras $C^*(E)$, it is now well known (see [14] for example) that a directed graph E has no loops if and only if its graph C^* -algebra $C^*(E)$ is an AF algebra. Moreover it is known [7] that every AF algebra is strong Morita equivalent to a graph algebra. The class of graph C^* -algebras was introduced in [14, 15] as a generalization of the Cuntz-Kreiger algebras [6] associated with finite matrices. The main benefit of working with graph algebras lies in the fact that many complex properties and structures of graph algebras can be explained in terms of conditions of graphs that are visible objects (see, for example, [2, 3, 14, 15, 12] among many others).

Besides the graph algebras, there have been various generalizations of Cuntz-Kreiger algebras, for example, the ultra-graph algebras [16] and the Exel-Laca algebras [9] are those generalizations which also include the C^* -algebras of row-finite graphs with no sinks. In [13], conditions for an AF algebra to be realized as a graph algebra, an Exel-Laca algebra, and an ultragraph algebra are given. Then in [8], it is proven that if a higher-rank graph algebra $C^*(\Lambda)$ is AF, the higher-rank graph Λ does not have an appropriate analogue of loop. The higher-rank graph algebras are of course another generalization of the Cuntz-Kreiger algebras.

Recently, a class of C^* -algebras $C^*(E, \mathcal{L}, \mathcal{B})$ associated to labeled graphs (E, \mathcal{L}) , more explicitly labeled spaces $(E, \mathcal{L}, \mathcal{B})$, has been introduced in [4] and studied in [5, 1, 10, 11]. We investigate in this paper the question of when a labeled graph C^* -algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is AF, where $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is a labeled space such that the accommodating set $\overline{\mathcal{E}^0}$ consisting of certain vertex subsets is closed under relative complements. For this, we first provide definitions of a loop and three possible types of exits in a labeled space. Then it is shown that if a labeled space $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has a loop α with an exit of any type, its associated C^* -algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has an infinite

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projection, which extends the same result that holds in graph algebras [14]. We also prove that if $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is an AF algebra, the labeled space $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has no loops. For the converse, a partial result is obtained under some extra conditions that $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is isomorphic to a graph algebra. A sufficient condition for $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ to be AF is given in terms of labeled path structure in Theorem 3.14 and various examples are analyzed.

2. PRELIMINARIES

2.1. Labeled spaces. We use notational conventions of [14] for graphs and graph C^* -algebras and of [5] for labeled spaces and their C^* -algebras. A *directed graph* is a quadruple $E = (E^0, E^1, r, s)$ consisting of a countable set of vertices E^0 , a countable set of edges E^1 , and the range, source maps $r_E, s_E : E^1 \rightarrow E^0$ (we often write r and s for r_E and s_E , respectively). If a vertex $v \in E^0$ emits (receives, respectively) no edges, v is called a *sink* (source, respectively). E_{sink}^0 denotes the set of all sinks of E and E^n denotes the set of all finite paths $\lambda = \lambda_1 \cdots \lambda_n$ of *length* n ($|\lambda| = n$), $(\lambda_i \in E^1, r(\lambda_i) = s(\lambda_{i+1}), 1 \leq i \leq n-1)$. We write $E^{\leq n}$ and $E^{\geq n}$ for the sets $\cup_{i=1}^n E^i$ and $\cup_{i=n}^\infty E^i$, respectively. The maps r and s naturally extend to $E^{\geq 0}$, where $r(v) = s(v) = v$ for $v \in E^0$. If a sequence of edges $\lambda_i \in E^1$ ($i \geq 1$) satisfies $r(\lambda_i) = s(\lambda_{i+1})$, one obtains an infinite path $\lambda_1 \lambda_2 \lambda_3 \cdots$ with the source $s(\lambda_1 \lambda_2 \lambda_3 \cdots) := s(\lambda_1)$ and by E^∞ we denote the set of all infinite paths. For a vertex subset $A \subset E^0$, A_{sink} denotes the sinks $A \cap E_{\text{sink}}^0$ in A , and for $\mathcal{B} \subset 2^{E^0}$, we simply denote the set $\{A_{\text{sink}} : A \in \mathcal{B}\}$ by $\mathcal{B}_{\text{sink}}$. Also with abuse of notation, for $\mathcal{B} \subset 2^{E^0}$ and $A \subset E_0$, we write

$$\mathcal{B} \cap A := \{B \in \mathcal{B} : B \subset A\}.$$

A *labeled graph* (E, \mathcal{L}) over a countable alphabet \mathcal{A} consists of a directed graph E and a *labeling map* $\mathcal{L} : E^1 \rightarrow \mathcal{A}$. We assume that \mathcal{L} is onto. Let \mathcal{A}^* and \mathcal{A}^∞ be the sets of all finite sequences (of length greater than or equal to 1) and infinite sequences with terms in \mathcal{A} , respectively. Then $\mathcal{L}(\lambda) := \mathcal{L}(\lambda_1) \cdots \mathcal{L}(\lambda_n) \in \mathcal{A}^*$ for $\lambda = \lambda_1 \cdots \lambda_n \in E^n$, and $\mathcal{L}(\delta) := \mathcal{L}(\delta_1) \mathcal{L}(\delta_2) \cdots \in \mathcal{L}(E^\infty) \subset \mathcal{A}^\infty$ for $\delta = \delta_1 \delta_2 \cdots \in E^\infty$. We use notation $\mathcal{L}^*(E) := \mathcal{L}(E^{\geq 1})$. For $\alpha = \alpha_1 \alpha_2 \cdots \alpha_{|\alpha|} \in \mathcal{L}^*(E)$, we denote the subsegment $\alpha_i \cdots \alpha_j$ of α by $\alpha_{[i,j]}$ for $1 \leq i \leq j \leq |\alpha|$. A subsegment of the form $\alpha_{[1,j]}$ is called an *initial path* of α . The *range* $r(\alpha)$ and *source* $s(\alpha)$ of a labeled path $\alpha \in \mathcal{L}^*(E)$ are subsets of E^0 defined by

$$\begin{aligned} r(\alpha) &= \{r(\lambda) : \lambda \in E^{\geq 1}, \mathcal{L}(\lambda) = \alpha\}, \\ s(\alpha) &= \{s(\lambda) : \lambda \in E^{\geq 1}, \mathcal{L}(\lambda) = \alpha\}. \end{aligned}$$

The *relative range* of $\alpha \in \mathcal{L}^*(E)$ with respect to $A \subset 2^{E^0}$ is defined to be

$$r(A, \alpha) = \{r(\lambda) : \lambda \in E^{\geq 1}, \mathcal{L}(\lambda) = \alpha, s(\lambda) \in A\}.$$

If $\mathcal{B} \subset 2^{E^0}$ is a collection of subsets of E^0 such that $r(A, \alpha) \in \mathcal{B}$ whenever $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$, \mathcal{B} is said to be *closed under relative ranges* for (E, \mathcal{L}) . We call \mathcal{B} an *accommodating set* for (E, \mathcal{L}) if it is closed under relative ranges, finite intersections

and unions and contains $r(\alpha)$ for all $\alpha \in \mathcal{L}^*(E)$. A set $A \in \mathcal{B}$ is called *minimal* (in \mathcal{B}) if A does not have any proper subset in \mathcal{B} .

If \mathcal{B} is accommodating for (E, \mathcal{L}) , the triple $(E, \mathcal{L}, \mathcal{B})$ is called a *labeled space*. For $A, B \in 2^{E^0}$ and $n \geq 1$, let

$$AE^n = \{\lambda \in E^n : s(\lambda) \in A\}, \quad E^n B = \{\lambda \in E^n : r(\lambda) \in B\},$$

and $AE^n B = AE^n \cap E^n B$. We write $E^n v$ for $E^n \{v\}$ and vE^n for $\{v\}E^n$, and will use notations like $AE^{\geq k}$ and vE^∞ which should have their obvious meaning. A labeled space $(E, \mathcal{L}, \mathcal{B})$ is said to be *set-finite* (*receiver set-finite*, respectively) if for every $A \in \mathcal{B}$ and $l \geq 1$ the set $\mathcal{L}(AE^l)$ ($\mathcal{L}(E^l A)$, respectively) is finite. A labeled space $(E, \mathcal{L}, \mathcal{B})$ is *weakly left-resolving* if

$$r(A, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha)$$

holds for all $A, B \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$. A labeled space (E, \mathcal{L}) is *left-resolving* if the map $\mathcal{L} : r^{-1}(v) \rightarrow \mathcal{A}$ is injective for each $v \in E^0$, and *label-finite* if $|\mathcal{L}^{-1}(a)| < \infty$ for each $a \in \mathcal{L}(E^1)$. If (E, \mathcal{L}) is left-resolving, then it is label-finite if and only if $r(a)$ is finite for all $a \in \mathcal{L}(E^1)$.

By $\Omega_0(E)$ we denote the set of all vertices of E that are not sources. For $v, w \in \Omega_0(E) \subset E^0$, we write $v \sim_l w$ if $\mathcal{L}(E^{\leq l} v) = \mathcal{L}(E^{\leq l} w)$ as in [5]. Then \sim_l is an equivalence relation on the set $\Omega_0(E)$. The equivalence class $[v]_l$ of v is called a *generalized vertex*. Let $\Omega_l(E) := \Omega_0(E) / \sim_l$ for $l \geq 1$. If $k > l$, $[v]_k \subset [v]_l$ is obvious and $[v]_l = \cup_{i=1}^m [v_i]_{l+1}$ for some vertices $v_1, \dots, v_m \in [v]_l$ ([5, Proposition 2.4]). If v is a source, we have $\mathcal{L}(E^{\leq l} v) = \emptyset$ for all $l \geq 1$, hence $[v]_l = [w]_l$ for all sources v, w and all $l \geq 1$.

Standing Assumptions. We assume that a labeled space $(E, \mathcal{L}, \mathcal{B})$ considered in this paper always satisfies the following:

- (i) $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving.
- (ii) $(E, \mathcal{L}, \mathcal{B})$ is set-finite and receiver set-finite.

Also we assume that if $v \in E^0$ is a sink, it is not a source. A labeled space $(E, \mathcal{L}, \mathcal{B})$ is said to be *finite* if there are only finitely many generalized vertices $[v]_l$ for each $l \geq 1$, or equivalently there are only finitely many labels.

We denote by \mathcal{E}^0 the smallest accommodating set for (E, \mathcal{L}) (cf. [4, p.108]);

$$\mathcal{E}^0 = \{\cup_{k=1}^m \cap_{i=1}^n r(\beta_{i,k}) : \beta_{i,k} \in \mathcal{L}^*(E)\},$$

and by $\mathcal{E}^{0,-}$ the smallest accommodating set containing $\mathcal{E}^- := \{r(\alpha) : \alpha \in \mathcal{L}^*(E)\} \cup \{\{v\} : v \text{ is a sink or a source}\}$.

If E has no sinks or sources, $\mathcal{E}^{0,-} = \mathcal{E}^0$ and every set in \mathcal{E}^0 can be expressed as a finite union of generalized vertices ([5, Remark 2.1 and Proposition 2.4.(ii)]);

$$\mathcal{E}^0 = \{\cup_{i=1}^n [v_i]_l : v_i \in \Omega_0(E), n, l \geq 1\}. \quad (1)$$

Generalized vertices $[v]_l$ are not always members of the accommodating set \mathcal{E}^0 but always the relative complements of sets in \mathcal{E}^0 , so $[v]_l = X_l(v) \setminus r(Y_l(v))$, where

$X_l(v)$, $Y_l(v)$ are given by

$$X_l(v) := \cap_{\alpha \in \mathcal{L}(E \leq^l v)} r(\alpha) \quad \text{and} \quad Y_l(v) := \cup_{w \in X_l(v)} \mathcal{L}(E \leq^l w) \setminus \mathcal{L}(E \leq^l v)$$

so that $X_l(v)$, $r(Y_l(v)) \in \mathcal{E}^0$ ([5, Proposition 2.4]). (Note that $[v]_l \cap r(Y_l(v)) = \emptyset$.)

The accommodating set \mathcal{E}^0 is not closed under relative complements, in general. On the other hand, in the construction of the C^* -algebra $C^*(E, \mathcal{L}, \mathcal{B})$ ([4, 5]) each nonempty set $A \in \mathcal{B}$ corresponds to a nonzero projection p_A in $C^*(E, \mathcal{L}, \mathcal{B})$ and if $A, B \in \mathcal{B}$ and $A \subset B$, their corresponding projections should satisfy $p_A \leq p_B$. Hence the projection $p_B - p_A$ belongs to $C^*(E, \mathcal{L}, \mathcal{B})$ and it seems reasonable to write $p_{B \setminus A}$ for $p_B - p_A$, which leads us to consider accommodating sets that are closed under relative complements.

Notation 2.1. Let (E, \mathcal{L}) be a labeled graph.

- (i) For a labeled space $(E, \mathcal{L}, \mathcal{B})$, we denote by $\overline{\mathcal{B}}$ (or $\overline{\mathcal{B}}(E, \mathcal{L})$) the smallest accommodating set that contains $\mathcal{B} \cup \mathcal{B}_{\text{sink}}$ and is closed under relative complements. The existence of $\overline{\mathcal{B}}$ follows clearly from considering the intersection of all those accommodating sets. $\overline{\mathcal{E}^0}$ will thus denote the smallest accommodating set that is closed under relative complements and contains the sets in $\overline{\mathcal{E}^0}_{\text{sink}} = \{A_{\text{sink}} : A \in \overline{\mathcal{E}^0}\}$.
- (ii) $\mathcal{L}^\#(E)$ will denote the union of all labeled paths $\mathcal{L}^*(E)$ and empty word ϵ , where ϵ is a symbol such that $r(\epsilon) = E^0$, $r(A, \epsilon) = A$ for all $A \subset E^0$.

Proposition 2.2. *Let (E, \mathcal{L}) be a labeled graph and $A \in \overline{\mathcal{E}^0}$. Then A is of the form*

$$A = (\cup_{i=1}^{n_1} [v_i]_l) \cup (\cup_{j=1}^{n_2} ([u_j]_l)_{\text{sink}}) \cup (\cup_{k=1}^{n_3} [w_k]_l \setminus ([w_k]_l)_{\text{sink}})$$

for some $v_i, u_j, w_k \in \Omega_0(E)$ and $l \geq 1$, $n_1, n_2, n_3 \geq 0$.

Proof. Let \mathcal{B} be the set of all such A 's. Then $\mathcal{B} \subset \overline{\mathcal{E}^0}$ is obvious since $\overline{\mathcal{E}^0}$ contains all generalized vertices. Now it suffices to show that \mathcal{B} is an accommodating set that is closed under relative complements. By [5, Proposition 2.4] or its proof, $r(\alpha) \in \mathcal{B}$ for all labeled paths $\alpha \in \mathcal{L}^*(E)$. It is easy to see that \mathcal{B} is closed under finite unions, finite intersections and relative complements.

In order to show that \mathcal{B} is closed under relative ranges, it suffices to see that $r([v]_l, \alpha) \in \mathcal{B}$ for $v \in \Omega_0(E)$ and $\alpha \in \mathcal{L}^*(E)$. Since $r([v]_l, \alpha) \cap r(r(Y_l(v)), \alpha) = r([v]_l \cap r(Y_l(v)), \alpha) = r(\emptyset, \alpha) = \emptyset$, we have

$$r([v]_l, \alpha) = r(X_l(v) \setminus r(Y_l(v)), \alpha) = r(X_l(v), \alpha) \setminus r(r(Y_l(v)), \alpha)$$

which belongs to \mathcal{B} since $r(X_l(v), \alpha)$, $r(r(Y_l(v)), \alpha) \in \mathcal{E}^0 \subset \mathcal{B}$ and \mathcal{B} is closed under relative complements. \square

2.2. Labeled graph C^* -algebras.

Definition 2.3. ([4, Definition 4.1] and [5, Remark 3.2]) Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labeled space such that $\overline{\mathcal{E}^0} \subset \mathcal{B}$. A *representation* of $(E, \mathcal{L}, \mathcal{B})$ consists of projections $\{p_A : A \in \mathcal{B}\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ such that for $A, B \in \mathcal{B}$ and $a, b \in \mathcal{A}$,

- (i) $p_\emptyset = 0$, $p_A p_B = p_{A \cap B}$, and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$,
- (ii) $p_A s_a = s_a p_{r(A,a)}$,
- (iii) $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$,
- (iv) for each $A \in \mathcal{B}$,

$$p_A = \sum_{a \in \mathcal{L}(AE^1)} s_a p_{r(A,a)} s_a^* + p_{A_{\text{sink}}}.$$

Remark 2.4. Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labeled space such that $\overline{\mathcal{E}^0} \subset \mathcal{B}$.

- (i) The proof of [4, Theorem 4.5] shows that for a weakly left-resolving labeled space $(E, \mathcal{L}, \mathcal{B})$, there exists a C^* -algebra $C^*(E, \mathcal{L}, \mathcal{B})$ generated by a universal representation $\{s_a, p_A\}$ of $(E, \mathcal{L}, \mathcal{B})$ (we need to modify the proof slightly, namely we should mod out the $*$ -algebra $k_{(E, \mathcal{L}, \mathcal{B})}$ by the ideal J generated by the elements $q_{A \cup B} - q_A - q_B + q_{A \cap B}$ and $q_A - \sum_{a \in \mathcal{L}(AE^1)} s_a q_{r(A,a)} s_a^* - q_{A_{\text{sink}}}$ for $A, B \in \mathcal{B}$). If $\{s_a, p_A\}$ is a universal representation of $(E, \mathcal{L}, \mathcal{B})$, we simply write $C^*(E, \mathcal{L}, \mathcal{B}) = C^*(s_a, p_A)$ and call $C^*(E, \mathcal{L}, \mathcal{B})$ the *labeled graph C^* -algebra* of a labeled space $(E, \mathcal{L}, \mathcal{B})$. Note that $s_a \neq 0$ and $p_A \neq 0$ for $a \in \mathcal{A}$ and $A \in \mathcal{B}$, $A \neq \emptyset$, and that $s_\alpha p_A s_\beta^* \neq 0$ if and only if $A \cap r(\alpha) \cap r(\beta) \neq \emptyset$. By Definition 2.3(iv) and [4, Lemma 4.4] saying that with $s_\alpha := p_\alpha$ for $\alpha \in \mathcal{E}^0$,

$$(s_\alpha p_A s_\beta^*)(s_\gamma p_B s_\delta^*) = \begin{cases} s_{\alpha\gamma'} p_{r(A, \gamma') \cap B} s_\delta^*, & \text{if } \gamma = \beta\gamma' \\ s_\alpha p_{A \cap r(B, \beta')} s_\delta^*, & \text{if } \beta = \gamma\beta' \\ s_\alpha p_{A \cap B} s_\delta^*, & \text{if } \beta = \gamma \\ 0, & \text{otherwise,} \end{cases}$$

for $\alpha, \beta, \gamma, \delta \in \mathcal{L}^\#(E)$ and $A, B \in \mathcal{B}$, it follows that we have

$$C^*(E, \mathcal{L}, \mathcal{B}) = \overline{\text{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}^\#(E), A \in \mathcal{B}\} \quad (2)$$

with the convention that s_ϵ denotes the unit of the multiplier algebra of $C^*(E, \mathcal{L}, \mathcal{B})$ [1]. It is observed in [11] that if E has no sinks nor sources,

$$C^*(E, \mathcal{L}, \mathcal{E}^{0,-}) \cong C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}).$$

- (ii) Universal property of $C^*(E, \mathcal{L}, \mathcal{B}) = C^*(s_a, p_A)$ defines a strongly continuous action $\gamma : \mathbb{T} \rightarrow \text{Aut}(C^*(E, \mathcal{L}, \mathcal{B}))$, called the *gauge action*, such that

$$\gamma_z(s_a) = z s_a \quad \text{and} \quad \gamma_z(p_A) = p_A$$

for $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{B}$.

- (iii) From Definition 2.3(iv), we have for each $n \geq 1$,

$$p_A = \sum_{\alpha \in \mathcal{L}(AE^n)} s_\alpha p_{r(A, \alpha)} s_\alpha^* + \sum_{\gamma \in \mathcal{L}(AE^{\leq n-1})} s_\gamma p_{r(A, \gamma)_{\text{sink}}} s_\gamma^*,$$

where $\sum_{\gamma \in \mathcal{L}(AE^0)} s_\gamma p_{r(A, \gamma)_{\text{sink}}} s_\gamma^* := p_{A_{\text{sink}}}$. In fact,

$$\begin{aligned}
p_A &= \sum_{a \in \mathcal{L}(AE^1)} s_a p_{r(A, a)} s_a^* + p_{A_{\text{sink}}} \\
&= \sum_{a \in \mathcal{L}(AE^1)} s_a \left(\sum_{b \in \mathcal{L}(r(A, a)E^1)} s_b p_{r(A, ab)} s_b^* + p_{r(A, a)_{\text{sink}}} \right) s_a^* + p_{A_{\text{sink}}} \\
&= \sum_{\gamma \in \mathcal{L}(AE^2)} s_\gamma p_{r(A, \gamma)} s_\gamma^* + \sum_{a \in \mathcal{L}(AE^1)} s_a p_{r(A, a)_{\text{sink}}} s_a^* + p_{A_{\text{sink}}} \\
&= \sum_{\gamma \in \mathcal{L}(AE^2)} s_\gamma \left(\sum_c s_c p_{r(A, \gamma c)} s_c^* + p_{r(A, \gamma)_{\text{sink}}} \right) s_\gamma^* \\
&\quad + \sum_{a \in \mathcal{L}(AE^1)} s_a p_{r(A, a)_{\text{sink}}} s_a^* + p_{A_{\text{sink}}} \\
&= \sum_{\alpha \in \mathcal{L}(AE^3)} s_\alpha p_{r(A, \alpha)} s_\alpha^* + \sum_{\gamma \in \mathcal{L}(AE^2)} s_\gamma p_{r(A, \gamma)_{\text{sink}}} s_\gamma^* \\
&\quad + \sum_{a \in \mathcal{L}(AE^1)} s_a p_{r(A, a)_{\text{sink}}} s_a^* + p_{A_{\text{sink}}} \\
&= \dots \\
&= \sum_{\alpha \in \mathcal{L}(AE^n)} s_\alpha p_{r(A, \alpha)} s_\alpha^* + \sum_{\gamma \in \mathcal{L}(AE^{\leq n-1})} s_\gamma p_{r(A, \gamma)_{\text{sink}}} s_\gamma^*.
\end{aligned}$$

(iv) Let $B \in \overline{\mathcal{E}^0}$ and I_B the ideal of $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ generated by the projection p_B . Then by (2), one can easily check that

$$I_B = \overline{\text{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}^\#(E), A \in \overline{\mathcal{E}^0} \cap r(\mathcal{L}(BE^{\geq 0}))\}. \quad (3)$$

3. AF LABELED GRAPH C^* -ALGEBRAS

To find conditions of a labeled space which arises an AF C^* -algebras, we define following generalized notions of loop.

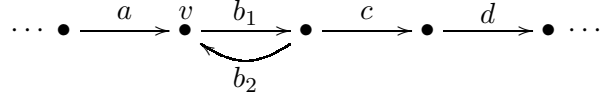
3.1. Loops and their exits.

Definition 3.1. Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space and $\alpha \in \mathcal{L}^*(E)$ a labeled path.

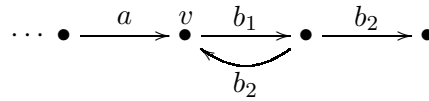
- (a) α is called a *generalized loop* at $A \in \mathcal{B}$ if $\alpha \in \mathcal{L}(AE^{\geq 1}A)$.
- (b) α is called a *loop* at $A \in \mathcal{B}$ if it is a generalized loop such that $A \subset r(A, \alpha)$.
- (c) A loop α at $A \in \mathcal{B}$ has an *exit* if one of the following holds:
 - (i) $\{\alpha_{[1, k]} : 1 \leq k \leq |\alpha|\} \subsetneq \mathcal{L}(AE^{\leq |\alpha|})$,
 - (ii) $r(A, \alpha_{[1, i]})_{\text{sink}} \neq \emptyset$ for some $i = 1, \dots, |\alpha|$,
 - (iii) $A \subsetneq r(A, \alpha)$.

Example 3.2. We give examples of labeled graphs with a loop that has an exit. In each of the following labeled graphs the accommodating set $\overline{\mathcal{E}^0}$ contains a singleton $\{v\}$ and a loop at it.

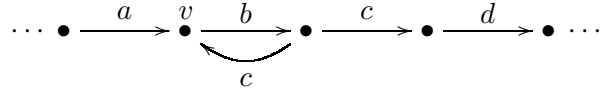
- (i) The loop $\alpha := b_1 b_2$ at $A := \{v\} \in \overline{\mathcal{E}^0}$ has an exit of type (i) of Definition 3.1(c) because $\mathcal{L}(AE^{\leq |\alpha|}) = \{b_1, b_1 b_2, bc\} \supsetneq \{b_1, b_1 b_2\}$.



- (ii) The loop $\alpha := b_1 b_2$ at $A := \{v\} \in \overline{\mathcal{E}^0}$ has an exit of type (ii) of Definition 3.1(c) because $r(A, \alpha)_{\text{sink}} \neq \emptyset$.



- (iii) The loop $\alpha := bc$ at $A := \{v\} \in \overline{\mathcal{E}^0}$ has an exit of type (iii) of Definition 3.1(c) because $A \subsetneq r(A, \alpha)$.



Remark 3.3. Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space and $A \in \mathcal{B}$.

- (i) A generalized loop α at a minimal set $A \in \mathcal{B}$ is necessarily a loop because $A \subset r(A, \alpha)$ follows from $A \cap r(A, \alpha) \neq \emptyset$ and the minimality of A . A labeled graph (E, \mathcal{L}) might have a (generalized) loop α even when the graph E itself has no loops at all as shown in Example 3.10.
- (ii) If α is a loop at A , then evidently $p_A \leq p_{r(A, \alpha)}$.

3.2. Existence of infinite projections.

Proposition 3.4. *Let (E, \mathcal{L}) be a labeled graph and α be a loop at $A \in \overline{\mathcal{E}^0}$ with an exit. Then $p_{r(A, \alpha)}$ is an infinite projection in $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$.*

Proof. If either $\mathcal{L}(AE^{\leq |\alpha|}) \supsetneq \{\alpha_{[1, k]} : 1 \leq k \leq |\alpha|\}$ or $r(A, \alpha_{[1, i]})_{\text{sink}} \neq \emptyset$ for some i , $1 \leq i \leq |\alpha|$, by Remark 2.4(iii) we have

$$p_A = \sum_{\beta \in \mathcal{L}(AE^{\leq |\alpha|})} s_\beta p_{r(A, \beta)} s_\beta^* + \sum_{|\gamma| \leq |\alpha| - 1} s_\gamma p_{r(A, \gamma)_{\text{sink}}} s_\gamma^* \geq s_\alpha p_{r(A, \alpha)} s_\alpha^*.$$

Thus $p_A > s_\alpha p_{r(A, \alpha)} s_\alpha^* \sim p_{r(A, \alpha)} \geq p_A$ and we see that the projection $p_{r(A, \alpha)}$ is infinite.

If $A \subsetneq r(A, \alpha)$, the projection $p_{r(A, \alpha)}$ is infinite because

$$p_{r(A, \alpha)} > p_A \geq s_\alpha p_{r(A, \alpha)} s_\alpha^* \sim p_{r(A, \alpha)}.$$

□

Remark 3.5. Let $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ be a labeled space and $\alpha_1, \dots, \alpha_n \in \mathcal{L}(AE^{|\alpha_1|})$ be labeled paths with the same length. If $A \subsetneq \bigcup_{i=1}^n r(A, \alpha_i)$, then the projection p_A is infinite: Set $A_1 := r(A, \alpha_1)$ and $A_i := r(A, \alpha_i) \setminus \bigcup_{j=1}^{i-1} r(A, \alpha_j)$, $i = 2, \dots, n$, so that the set $\bigcup_{i=1}^n r(A, \alpha_i) = \bigcup_{i=1}^n A_i$ is the union of disjoint A_i 's. Then

$$p_A \geq \sum_{i=1}^n s_{\alpha_i} p_{r(A, \alpha_i)} s_{\alpha_i}^* \geq \sum_{i=1}^n s_{\alpha_i} p_{A_i} s_{\alpha_i}^*$$

and the projection $\sum_{i=1}^n s_{\alpha_i} p_{A_i} s_{\alpha_i}^*$ is equivalent to $\sum_{i=1}^n p_{A_i} = p_{\bigcup A_i} = p_{\bigcup_{i=1}^n r(A, \alpha_i)}$. In fact, with $u := \sum_{i=1}^n s_{\alpha_i} p_{A_i}$ one has $uu^* = \sum_{i=1}^n s_{\alpha_i} p_{A_i} s_{\alpha_i}^*$ and $u^*u = \sum_{i=1}^n p_{A_i}$. This shows that p_A is infinite since $p_{\bigcup_{i=1}^n r(A, \alpha_i)} \succeq p_A$.

Proposition 3.6. *Let $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ be a labeled space such that $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has no infinite projections. Let a set $A \in \overline{\mathcal{E}^0}$ admit a loop. Then there exists a loop $\alpha \in \mathcal{L}^*(E)$ such that $A = r(A, \alpha)$ and $\mathcal{L}(AE^{\geq 1}A) = \{\alpha^k\}_{k \geq 1}$. Moreover every path $\beta \in \mathcal{L}(AE^{\geq 1})$ is of the form $\beta = \alpha^k \alpha'$ for some $k \geq 0$ and an initial path α' of α .*

Proof. Choose a loop $\alpha \in \mathcal{L}(AE^{\geq 1}A)$ with the smallest length; $|\alpha| \leq |\gamma|$ for all loops $\gamma \in \mathcal{L}(AE^{\geq 1}A)$. Since $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has no infinite projections, α does not have an exit by Proposition 3.4, hence $A = r(A, \alpha)$ and $\mathcal{L}(AE^{\leq |\alpha|}) = \{\alpha_{[1, k]}\} : 1 \leq k \leq |\alpha|\}$. Thus $\alpha^k \in \mathcal{L}(AE^{\geq 1}A)$ for all $k \geq 1$.

Now suppose that there exists a path $\beta \in \mathcal{L}(AE^{\geq 1})$ with $|\beta| > |\alpha|$ that is not of the form $\alpha^k \alpha'$. Then from $\mathcal{L}(AE^{|\alpha|}) = \{\alpha\}$ we can write $\beta = \alpha \beta'$ for a path β' . But then β' must be an initial path of α or of the form $\alpha \beta''$ for some path β'' . Applying the argument repeatedly, finally we end up with $\beta = \alpha^k \alpha'$ for some $k \geq 1$ and an initial path α' of α . \square

3.3. AF labeled graph C^* -algebras. We give a necessary condition and also a sufficient condition for a labeled graph C^* -algebra to be AF, and examine various examples.

Theorem 3.7. *Let (E, \mathcal{L}) be a labeled graph. If $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is an AF algebra, the labeled space $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has no loops.*

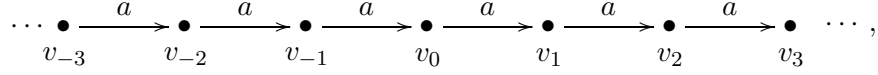
Proof. Suppose, for contradiction, that $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has a loop α at $A \in \overline{\mathcal{E}^0}$. By Proposition 3.4, $A = r(A, \alpha)$, and so $p_A s_\alpha = s_\alpha p_{r(A, \alpha)} = s_\alpha p_A$. Set $U = s_\alpha p_A$. Then

$$p_A = U^* U \sim U U^* = s_\alpha p_A s_\alpha^* = s_\alpha p_{r(A, \alpha)} s_\alpha^* \leq p_A.$$

Since p_A is a finite projection, it follows that U is a unitary of the unital hereditary subalgebra $p_A C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) p_A$. Since $\gamma_z(p_A) = p_A$ for any $z \in \mathbb{T}$, the algebra $p_A C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) p_A$ admits an action (the restriction of the gauge action γ on $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$). Then the fact that $\gamma_z(U) = \gamma_z(s_\alpha) p_A = z^{|\alpha|} U$ shows that U is not in the unitary path connected component of the unit p_A ([8, Proposition 3.9]), which is a contradiction to the assumption that $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ (hence any nonzero hereditary subalgebra) is an AF algebra. \square

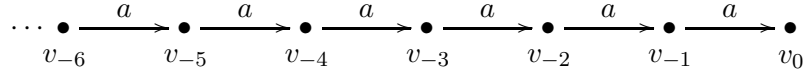
In Example 3.8(iii) below, we see that the converse of Theorem 3.7 may not be true, in general.

Example 3.8. (i) For the following labeled graph (E, \mathcal{L})



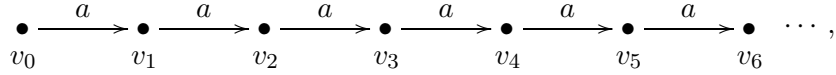
we have $\overline{\mathcal{E}^0} = \{r(a)\} = \{E^0\}$ and the path a is a loop at $r(a)$. By Theorem 3.7, $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) := C^*(s_a, p_A)$ is not AF. Actually $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) \cong C(\mathbb{T})$ is the universal C^* -algebra generated by a unitary s_a .

(ii) For the following labeled graph (E, \mathcal{L})



$\overline{\mathcal{E}^0}$ consists of three sets $r(a) = E^0$, $r(a)_{\text{sink}} = \{v_0\}$, and $A := r(A) \setminus r(a)_{\text{sink}} = \{v_{-1}, v_{-2}, \dots\}$. Since $A \subsetneq r(A, a)$, the loop a at A has an exit and $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ contains an infinite projection by Proposition .

(iii) If (E, \mathcal{L}) is as follows



it is not hard to see that $\overline{\mathcal{E}^0}$ consists of all finite sets F with $v_0 \notin F$ and all sets of the form $F \cup \{v_k, v_{k+1}, \dots\}$ for some $k \geq 1$. It is also easy to see that every set $A \in \overline{\mathcal{E}^0}$ containing at least two vertices always admits a generalized loop. But there does not exist a loop at any set $A \in \overline{\mathcal{E}^0}$, nevertheless we shall see that $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ contains an infinite projection and so is not AF. Let $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) = C^*(p_A, s_a)$. Then for $B := r(a)$, we have $r(B, a) \subsetneq B$, and a similar argument as in (ii) shows that the projection p_B is infinite; $p_{r(a)} = s_a p_{r(a), a} s_a^* = s_a p_{r(a^2)} s_a^* \sim p_{r(a^2)} < p_{r(a)}$.

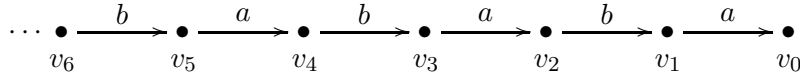
The C^* -algebra $C^*(p_A, s_a)$ is unital with the unit $s_a s_a^*$; $(s_a s_a^*) p_A = s_a p_{r(A, a)} s_a^* = p_A$ for all $A \in \overline{\mathcal{E}^0}$ and $(s_a s_a^*) s_a = s_a = s_a p_{r(a)} = s_a p_{r(a)} (s_a s_a^*) = s_a (s_a s_a^*)$. Also we have $s_a s_a^* \not\geq p_{r(a)} = s_a^* s_a$ since $s_a s_a^* \geq s_a p_{\{v_1\}} s_a^* (\neq 0)$ and $(s_a p_{\{v_1\}} s_a^*) p_A = s_a p_{\{v_1\}} p_{r(A, a)} s_a^* = s_a p_{\{v_1\} \cap r(A, a)} s_a^* = 0$ for all $A \in \overline{\mathcal{E}^0}$. Moreover every projection p_A belongs to the $*$ -algebra generated by s_a . Therefore $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is the universal C^* -algebra generated by a proper coisometry s_a , and thus $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is the Toeplitz algebra. The ideal $I_{\{v_1\}}$ generated by the projection $p_{\{v_1\}}$ is in fact isomorphic to the C^* -algebra of compact operators on an infinite dimensional separable Hilbert space as $I_{\{v_1\}} = \overline{\text{span}}\{s_a^m p_{\{v_i\}} (s_a^*)^n : m, n \geq 0 \text{ and } i \geq 1\}$ (see (3)). The quotient algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})/I_{\{v_1\}}$ is isomorphic to $C(\mathbb{T})$.

Remark 3.9. The first part of Example 3.8(iii) (or basically Remark 2.4) actually proves that if there exists a path $\alpha \in \mathcal{L}^*(E)$ and $k \geq 1$ such that

- (i) $r(\alpha^k) \subsetneq r(\alpha)$,
- (ii) $\mathcal{L}(r(\alpha)E^{|\alpha|^k}) = \{\alpha^k\}$, and
- (iii) $r(r(\alpha), \gamma)_{\text{sink}} = \emptyset$ for all $\gamma \in \mathcal{L}(E^{<|\alpha|^k})$,

then the C^* -algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has an infinite projection, $p_{r(\alpha)}$.

Example 3.10. The accommodating set $\overline{\mathcal{E}^0}$ of the following labeled graph



is $\overline{\mathcal{E}^0} = \{\emptyset, r(a), r(a)_{\text{sink}}, r(a) \setminus r(a)_{\text{sink}}, r(b), E^0\}$, where $r(a) = r(a)_{\text{sink}} \cup (r(a) \setminus r(a)_{\text{sink}}) = A_0 \cup A_2$ and $r(b) = A_1$, with three minimal sets

$$A_0 = \{v_0\}, \quad A_1 := \{v_1, v_3, v_5, \dots\}, \quad A_2 = \{v_2, v_4, v_6, \dots\}.$$

Here A_1 and A_2 have loops as $\mathcal{L}(A_1 E^{\geq 1} A_1) = \{(ab)^k : k \geq 1\}$ and $\mathcal{L}(A_2 E^{\geq 1} A_2) = \{(ba)^k : k \geq 1\}$. Thus by Theorem 3.7, the C^* -algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is not AF.

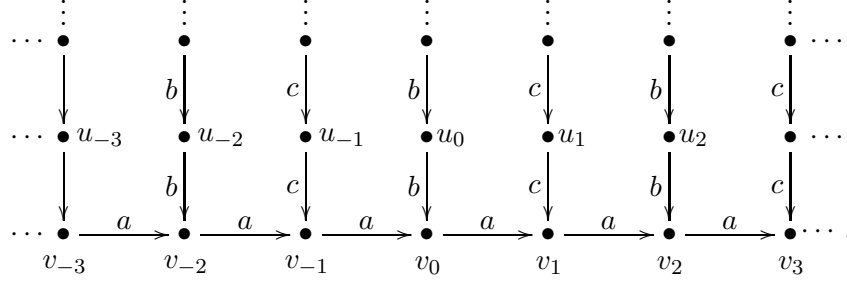
For a labeled graph (E, \mathcal{L}_E) , $v \sim w$ if and only if $v \sim_l w$ for all $l \geq 1$ defines an equivalence relation on E^0 . We denote the equivalence class of $v \in E^0$ by $[v]_\infty$. If (E, \mathcal{L}_E) has no sinks or sources, there exists a labeled graph (F, \mathcal{L}_F) called the *merged labeled graph* of (E, \mathcal{L}_E) with vertices $\{[v]_\infty : v \in E^0\}$ and edges $\{e_\lambda : \lambda \in E^1\}$, where e_λ is a path with $s_F(e_\lambda) = [s(\lambda)]_\infty$, $r_F(e_\lambda) = [r(\lambda)]_\infty$, and $\mathcal{L}_F(e_\lambda) = \mathcal{L}_E(\lambda)$. It is known in [11, Theorem 6.10] that if $[v]_\infty \in \overline{\mathcal{E}^0}$ for all $v \in E^0$, then $\{[v]_\infty\} \in \overline{\mathcal{F}^0}$ for all $[v]_\infty \in F^0$ and moreover $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) \cong C^*(F, \mathcal{L}, \overline{\mathcal{F}^0})$. For (E, \mathcal{L}_E) with sinks or sources, the result of [11, Theorem 6.10] can be proved to be true without significant modification; $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ and $C^*(F, \mathcal{L}, \overline{\mathcal{F}^0})$ are isomorphic whenever $[v]_\infty \in \overline{\mathcal{E}^0}$ for all $v \in E^0$. If E has no sinks and every generalized vertex of $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is a finite union of minimal sets, then $[v]_\infty \in \overline{\mathcal{E}^0}$ for all $v \in \Omega_0(E)$.

Proposition 3.11. *Let (E, \mathcal{L}_E) be a row-finite left-resolving labeled graph with no sinks or sources such that every generalized vertex is a finite union of minimal sets in $\overline{\mathcal{E}^0}$. Then $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is isomorphic to the graph C^* -algebra $C^*(F)$ associated to the merged labeled graph (F, \mathcal{L}_F) of (E, \mathcal{L}_E) . Moreover $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is an AF algebra if and only if every minimal set of $\overline{\mathcal{E}^0}$ admits no loops.*

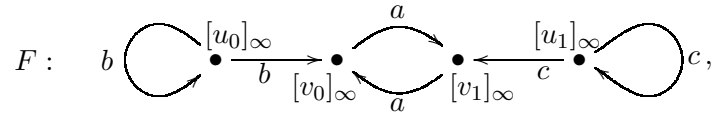
Proof. For $a \in \mathcal{L}(E^1)$, we can write $r(a) = \cup_{i=1}^n [w_i]_{l_i}$ for some minimal sets $[w_i]_{l_i} = [w_i]_\infty$ for each i . Then $r_F(a) = [r(a)]_\infty := \{[w]_\infty : w \in r(a)\} = \{[w_1]_\infty, \dots, [w_n]_\infty\}$, which means that (F, \mathcal{L}_F) is label-finite because (E, \mathcal{L}_E) is left-resolving. Thus by [11, Thm 6.9] and [4, Thm 6.6], $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) \cong C^*(F, \mathcal{L}_F, \overline{\mathcal{F}^0}) \cong C^*(F)$.

Now suppose that there is no loop at any minimal set $[v]_\infty$ in $\overline{\mathcal{E}^0}$. Then F has no loops at any of its vertex and $C^*(F)$ is an AF algebra. The converse was proved in Theorem 3.7. \square

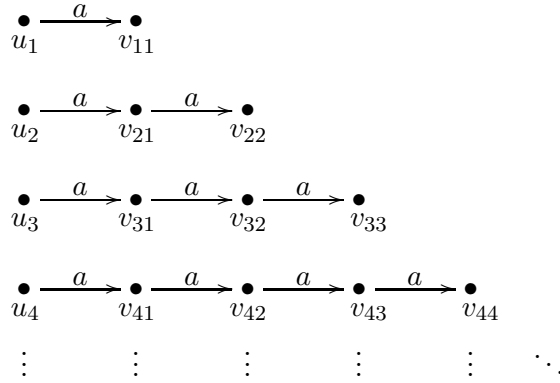
Example 3.12. In the following labeled graph (E, \mathcal{L})



the path $\alpha := a^2$ is a loop at $\{v_{2k} : k \in \mathbb{Z}\}$ and $\{v_{2k+1} : k \in \mathbb{Z}\}$ respectively. By Theorem 3.7, the C^* -algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is not AF. In fact, $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is isomorphic to $C^*(F)$ associated to the underlying graph F of the merged labeled graph (F, \mathcal{L}_F) by Proposition 3.11.



Example 3.13. The following labeled graph (E, \mathcal{L}) does not contain any infinite paths ($a^\infty \notin \mathcal{L}(E^\infty)$), but it contains finite paths a^n of any length n for all $n \geq 1$.



Note that each finite path a^n is not a loop at any $A \in \overline{\mathcal{E}^0}$ but it is a generalized loop at $r(a^k)$ for all $k \geq 1$, and $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has the generalized vertices as follows:

$$\begin{aligned}
 [v_{ij}]_k &= \begin{cases} r(a^k), & \text{if } 1 \leq k \leq j \\ r(a^j) \setminus r(a^{j+1}), & \text{if } 1 \leq j < k \end{cases} \\
 ([v_{ij}]_k)_{\text{sink}} &= \begin{cases} \{v_{mm} : m \geq k\}, & \text{if } 1 \leq k \leq j \\ \{v_{jj}\}, & \text{if } 1 \leq j < k \end{cases} \\
 [v_{ij}]_k \setminus ([v_{ij}]_k)_{\text{sink}} &= \begin{cases} \{v_{mn} : m > n \geq k\}, & \text{if } 1 \leq k \leq j \\ \{v_{mj} : m \geq j\}, & \text{if } 1 \leq j < k. \end{cases}
 \end{aligned}$$

It then follows that every $A \in \overline{\mathcal{E}^0}$ is a finite union of these sets.

Let J be the ideal of $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ generated by the projection $p_{[v_{11}]_2}$. Then (3) shows that

$$J = \overline{\text{span}}\{s_a^m p_B s_a^{*n} : B \in [v_{kk}]_{k+1} \cap \overline{\mathcal{E}^0}, m, n \geq 0, k \geq 1\}.$$

From $p_{r(a)} - p_{r(a^2)} = p_{r(a) \setminus r(a^2)} = p_{[v_{11}]_2} \in J$, we have

$$s_a + J = s_a p_{r(a)} + J = p_{r(a)} s_a + J.$$

Thus $s_a p_{r(a)} + J$ is a unitary of the unital hereditary subalgebra (with unit $p_{r(a)} + J$) of the quotient algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})/J$. The ideal J is obviously invariant under the gauge action $\gamma : \mathbb{T} \rightarrow \text{Aut}(C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}))$. Hence there exists an induced action $\gamma : \mathbb{T} \rightarrow \text{Aut}(C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})/J)$ such that $\gamma_z(s_a p_{r(a)} + J) = z(s_a p_{r(a)} + J)$ for $z \in \mathbb{T}$. Thus the unitary $s_a p_{r(a)} + J$ does not belong to the unitary path connected component of the unit of the hereditary subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})/J$, which implies as in the proof of Theorem 3.7 that $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})/J$ and hence $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is not AF.

For convenience, we use the notation $A_1 E^{\geq 1} A_2 \cdots E^{\geq 1} A_{n+1}$ to denote the set

$$\{x = x_1 x_2 \cdots x_n \in E^{\geq 1} : x_k \in A_k E^{\geq 1} A_{k+1}, 1 \leq k \leq n\}.$$

Theorem 3.14. *Let (E, \mathcal{L}) be a labeled graph. Assume that if A_1, A_2, \dots is a sequence of sets in $\overline{\mathcal{E}^0}$ such that*

$$A_1 E^{\geq 1} A_2 E^{\geq 1} A_3 \cdots E^{\geq 1} A_n \neq \emptyset$$

for all $n \geq 1$, the set $\{A_1, A_2, \dots\}$ is infinite. Then $C^(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is AF.*

Proof. Let $F := \{s_{\alpha_i} p_{A_i} s_{\beta_i}^* : A_i \subset r(\alpha_i) \cap r(\beta_i), i = 1, \dots, N\}$ be a finite set in the C^* -algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) = C^*(s_a, p_A)$ with $F = F^*$. We shall show that F generates a finite dimensional C^* -algebra. Set $K := \max\{|\alpha_i|, |\beta_i| : i = 1, \dots, N\}$. By Remark 2.4(i), we have

$$(s_{\alpha_i} p_{A_i} s_{\beta_i}^*)(s_{\alpha_j} p_{A_j} s_{\beta_j}^*) = \begin{cases} s_{\alpha_i \gamma'} p_{r(A_i, \gamma') \cap A_j} s_{\beta_j}^*, & \text{if } \alpha_j = \beta_i \gamma' \\ s_{\alpha_i} p_{A_i \cap r(A_j, \beta')} s_{\beta_j \beta'}^*, & \text{if } \beta_i = \alpha_j \beta' \\ s_{\alpha_i} p_{A_i \cap A_j} s_{\beta_j}^*, & \text{if } \beta_i = \alpha_j \\ 0, & \text{otherwise,} \end{cases}$$

and so if, for example, $\alpha_j = \beta_i \gamma'$ and $\alpha_k = \beta_j \gamma''$, we get

$$\begin{aligned} (s_{\alpha_i} p_{A_i} s_{\beta_i}^*)(s_{\alpha_j} p_{A_j} s_{\beta_j}^*)(s_{\alpha_k} p_{A_k} s_{\beta_k}^*) &= (s_{\alpha_i \gamma'} p_{r(A_i, \gamma') \cap A_j} s_{\beta_j}^*)(s_{\alpha_k} p_{A_k} s_{\beta_k}^*) \\ &= s_{\alpha_i \gamma' \gamma''} p_{r(r(A_i, \gamma') \cap A_j, \gamma'') \cap A_k} s_{\beta_k}^*. \end{aligned}$$

Here note that $\gamma' \gamma''$ belongs to $\mathcal{L}(A_i E^{|\gamma'|} A_j E^{|\gamma''|} A_k)$ and

$$r(r(A_i, \gamma') \cap A_j, \gamma'') \cap A_k = r(A_i \gamma' A_j \gamma'' A_k) \in r(\mathcal{L}(A_i E^{\leq K} A_j E^{\leq K} A_k)).$$

If we go one step further with $\beta_k = \alpha_l \beta'$, we obtain that

$$\begin{aligned} & (s_{\alpha_i} p_{A_i} s_{\beta_i}^*) (s_{\alpha_j} p_{A_j} s_{\beta_j}^*) (s_{\alpha_k} p_{A_k} s_{\beta_k}^*) (s_{\alpha_l} p_{A_l} s_{\beta_l}^*) \\ &= (s_{\alpha_i \gamma' \gamma''} p_{r(A_i, \gamma') \cap A_j, \gamma''} \cap A_k s_{\beta_k}^*) (s_{\alpha_l} p_{A_l} s_{\beta_l}^*) \\ &= s_{\alpha_i \gamma' \gamma''} p_{r(A_i, \gamma') \cap A_j, \gamma''} \cap A_k \cap r(A_l, \beta') s_{\beta_l \beta'}^* \end{aligned}$$

is nonzero only when $\gamma' \gamma'' \in \mathcal{L}(A_i E^{|\gamma'|} A_j E^{|\gamma''|} A_k)$ and $\beta' \in \mathcal{L}(A_l E^{|\beta'|} A_k)$, and moreover if this is the case, the set

$$r(r(A_i, \gamma') \cap A_j, \gamma'') \cap A_k \cap r(A_l, \beta') = r(A_i \gamma' A_j \gamma'' A_k) \cap r(A_l \beta' A_k)$$

can be written $r(\delta_1) \cap r(\delta_2)$ for some δ_1, δ_2 with $\delta_1 \in \mathcal{L}(A_i E^{\leq K} A_j E^{\leq K} A_k)$ and $\delta_2 \in \mathcal{L}(A_l E^{\leq K} A_k)$. Repeating the process we see that multiplying any finite elements from F gives an element of the form $s_{\alpha_i \mu} p_A s_{\beta_j \nu}^*$, where A is a finite intersection of some sets in

$$A(F) := \{r(\delta) : \delta \in \mathcal{L}(A_{i_1} E^{\leq K} A_{i_2} \cdots E^{\leq K} A_{i_n}), i_j \in \{1, \dots, N\}, n \geq 1\},$$

and μ and ν are paths in

$$\mathcal{L}(F) := \cup_{m \geq 1} \mathcal{L}(A_{i_1} E^{\leq K} A_{i_2} \cdots E^{\leq K} A_{i_m}).$$

In order to consider the $*$ -algebra generated by F we need to take into account every set that appears as a finite intersection of sets in $A(F)$ and labeled paths in $\mathcal{L}(F)$. Since $A_{i_j} \in \{A_1, \dots, A_N\}$ for all i_j , by our assumption, the set $\mathcal{L}(F)$ can not be infinite, thus $\mathcal{L}(F) = \cup_{m=1}^M \mathcal{L}(A_{i_1} E^{\leq K} A_{i_2} \cdots E^{\leq K} A_{i_m})$ for some $M \geq 1$. Then the set of all finite intersections of sets in

$$A(F) := \{r(\delta) : \delta \in \mathcal{L}(A_{i_1} E^{\leq K} A_{i_2} \cdots E^{\leq K} A_{i_n}), i_j \in \{1, \dots, N\}, 1 \leq n \leq M\}$$

is also finite. Consequently the $*$ -algebra generated by F is equal to the following finite dimensional $*$ -algebra

$$\overline{\text{span}}\{s_{\alpha_i \mu} p_A s_{\beta_j \nu}^* : A = \cap B_k, B_k \in A(F), \mu, \nu \in \mathcal{L}(F), 1 \leq i, j \leq N\}.$$

□

3.4. Non-AF labeled graph C^* -algebras. If $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ satisfies the assumption of Theorem 3.14, for each $\alpha \in \mathcal{L}^*(E)$ there is an $n \geq 1$ such that $\alpha^n \notin \mathcal{L}^*(E)$ since $\alpha^n \in \mathcal{L}^*(E)$ implies $\alpha^{n-1} \in \mathcal{L}(r(a) E^{\geq 1} r(a) \cdots E^{\geq 1} r(a))$. It would be interesting to know whether $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is not AF whenever there is a path $\alpha \in \mathcal{L}^*(E)$ such that $\alpha^n \in \mathcal{L}^*(E)$ for all n .

For a labeled graph C^* -algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) = C^*(s_a, p_A)$ and a set $A \in \overline{\mathcal{E}^0}$, we denote by I_A the ideal of $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ generated by the projection p_A as before.

Lemma 3.15. *Let $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) = C^*(s_a, p_A)$ be the C^* -algebra of a labeled graph (E, \mathcal{L}) with no sinks or sources. For $A, B \in \overline{\mathcal{E}^0}$, we have $p_A \in I_B$ if and only if there exist an $N \geq 1$ and finitely many paths $\{\mu_i\}_{i=1}^N$ in $\mathcal{L}(B E^{\geq 0})$ such that*

$$\cup_{|\beta|=N} r(A, \beta) \subset \cup_{i=1}^N r(B, \mu_i).$$

Proof. If $p_A \in I_B$, we can approximate p_A , within $\varepsilon > 0$ small enough, by an element $\sum_{i=1}^n c_i s_{\beta_i} p_{r(B, \mu_i)} s_{\gamma_i}^* \in I_B$, where $c_i \in \mathbb{C}$, $\beta_i, \gamma_i \in \mathcal{L}(AE^{\geq 0})$, and $\mu_i \in \mathcal{L}(BE^{\geq 0})$ for $1 \leq i \leq n$. We assume $(\beta_i, \mu_i, \gamma_i) \neq (\beta_j, \mu_j, \gamma_j)$ if $i \neq j$. Considering the image of $X := p_A - \sum_{i=1}^n c_i s_{\beta_i} p_{r(B, \mu_i)} s_{\gamma_i}^*$ under the conditional expectation onto the AF core (the fixed point algebra the gauge action), we may assume that $|\beta_i| = |\gamma_i|$ for all i . Moreover, since (E, \mathcal{L}) has no sinks, we can also assume that $|\beta_i| = |\beta_1|$ for all i . Put $N := |\beta_1|$, $1 \leq i \leq n$. From $p_A = \sum_{|\beta|=N} s_{\beta} p_{r(A, \beta)} s_{\beta}^*$, we have

$$\|X\| = \left\| \sum_{|\beta|=N} s_{\beta} p_{r(A, \beta)} s_{\beta}^* - \sum_{i=1}^n c_i s_{\beta_i} p_{r(B, \mu_i)} s_{\gamma_i}^* \right\| < \varepsilon.$$

If $r(A, \beta) \not\subset \cup_{i=1}^n r(B, \mu_i)$ for some $\beta \in \mathcal{L}(AE^N)$, that is, $A' := r(A, \beta) \setminus \cup_{i=1}^n r(B, \mu_i) \neq \emptyset$, one obtains a contradiction, $\varepsilon > \|p_{A'}(s_{\beta}^* X s_{\beta}) p_{A'}\| = \|p_{A'}\| = 1$.

For the reverse inclusion, it is enough to note that $p_{\cup_{i=1}^n r(B, \mu_i)} \in I_B$ (see [11, Lemma 3.5]). \square

Theorem 3.16. *Let $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) = C^*(s_a, p_A)$ be the C^* -algebra of a labeled graph (E, \mathcal{L}) and let $\alpha \in \mathcal{L}^*(E)$ satisfy $\alpha^n \in \mathcal{L}^*(E)$ for all $n \geq 1$. If $p_{r(\alpha^m)}$ does not belong to the ideal generated by a projection $p_{r(\alpha^m) \setminus r(\alpha^{m+1})}$ for some $m \geq 1$, then $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is not AF.*

Proof. Let $A_m := r(\alpha^m) \setminus r(\alpha^{m+1})$. Then $\{I_{A_m}\}_{m=1}^{\infty}$ is a decreasing sequence of ideals because the generator $p_{r(\alpha^{m+1}) \setminus r(\alpha^{m+2})}$ of $I_{A_{m+1}}$ belongs to I_{A_m} , in fact, the projection $p_{r(\alpha^{m+1}) \setminus r(\alpha^{m+2})}$ is equal to $p_{r(r(\alpha^m) \setminus r(\alpha^{m+1}), \alpha)} = s_{\alpha}^* s_{\alpha} p_{r(r(\alpha^m) \setminus r(\alpha^{m+1}), \alpha)} = s_{\alpha}^* p_{r(\alpha^m) \setminus r(\alpha^{m+1})} s_{\alpha}$ which belongs to I_{A_m} . We first show the following claim.

Claim: If $p_{r(\alpha)}$ does not belong to the ideal generated by $p_{r(\alpha) \setminus r(\alpha^2)}$, then the C^* -algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is not AF.

To prove the claim, it is enough to show that the quotient algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})/I_{A_1}$ is not AF. Note that $p_{r(\alpha)} + I_{A_1} = p_{r(\alpha^2)} + I_{A_1}$ is a nonzero projection in the quotient algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})/I_{A_1}$ and that

$$I_{A_1} = \overline{\text{span}}\{s_{\beta} p_B s_{\gamma}^* : \beta, \gamma \in \mathcal{L}(E^{\geq 0}) \text{ and } B \in r(\mathcal{L}(A_1 E^{\geq 0})) \cap \overline{\mathcal{E}^0}\}.$$

If $s_{\alpha}^* s_{\alpha} + I_{A_1} = s_{\alpha} s_{\alpha}^* + I_{A_1}$, the unital hereditary subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})/I_{A_1}$ with the unit projection $p_{r(\alpha)} + I_{A_1}$ is not AF since it contains a unitary $s_{\alpha} + I_{A_1}$ satisfying $\gamma_z(s_{\alpha} + I_{A_1}) = z^{|\alpha|}(s_{\alpha} + I_{A_1})$ for each $z \in \mathbb{C}$. Thus the hereditary subalgebra (hence $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$) is not an AF algebra. (The fact that $s_{\alpha} + I_{A_1}$ belongs to the hereditary subalgebra follows from $p_{r(\alpha)} s_{\alpha} + I_{A_1} = s_{\alpha} p_{r(\alpha^2)} + I_{A_1} = s_{\alpha} p_{r(\alpha)} + I_{A_1} = s_{\alpha} + I_{A_1}$.) If $s_{\alpha}^* s_{\alpha} + I_{A_1} \neq s_{\alpha} s_{\alpha}^* + I_{A_1}$, then $s_{\alpha}^* s_{\alpha} + I_{A_1} = p_{r(\alpha)} + I_{A_1} \geq s_{\alpha} p_{r(\alpha^2)} s_{\alpha}^* + I_{A_1} = s_{\alpha} p_{r(\alpha)} s_{\alpha}^* + I_{A_1} = s_{\alpha} s_{\alpha}^* + I_{A_1}$ and this shows that $s_{\alpha}^* s_{\alpha} + I_{A_1} \not\leq s_{\alpha} s_{\alpha}^* + I_{A_1}$. Thus the projection $s_{\alpha}^* s_{\alpha} + I_{A_1}$ is infinite, and the quotient algebra is not AF as claimed.

Now suppose that $p_{r(\alpha^m)} \notin I_{A_m}$ for some $m \geq 2$. Since $\delta := \alpha^m$ satisfies $\delta^n \in \mathcal{L}^*(E)$ for all n , by the above claim, we only need to show that $p_{r(\delta)}$ does not belong to the ideal, say J , generated by the projection $p_{r(\delta) \setminus r(\delta^2)} = p_{r(\alpha^m) \setminus r(\alpha^{2m})}$. For this,

assuming $p_{r(\delta)} \in J$ we have from Lemma 3.15 that there exist an $N \geq 1$ and paths $\{\mu_j\}_{j=1}^n$ such that

$$r(r(\delta), \beta) \subset \cup_{i=1}^n r(r(\alpha^m) \setminus r(\alpha^{2m}), \mu_i)$$

for all $\beta \in \mathcal{L}(r(\delta)E^N)$. Since each set $r(r(\alpha^m) \setminus r(\alpha^{2m}), \mu_i)$ coincides with

$$\cup_{j=0}^{m-1} r(r(\alpha^{m+j}) \setminus r(\alpha^{m+j+1}), \mu_i) = \cup_{j=0}^{m-1} r(r(\alpha^m) \setminus r(\alpha^{m+1}), \alpha^j \mu_i),$$

we can write the set $\cup_{i=1}^n r(r(\alpha^m) \setminus r(\alpha^{2m}), \mu_i)$ as $\cup_{j=1}^{n'} r(r(\alpha^m) \setminus r(\alpha^{m+1}), \mu'_j)$ for some finitely many paths μ'_j which is of the form $\alpha^l \mu_i$. This means that $p_{r(\alpha^m)} = p_{r(\delta)} \in I_{A_m}$ again by Lemma 3.15, which is a contradiction. \square

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DEPARTMENT OF MATHEMATICAL SCIENCES AND RESEARCH INSTITUTE OF MATHEMATICS,
SEOUL NATIONAL UNIVERSITY, SEOUL, 151-747, KOREA

E-mail address: `jajeong@snu.ac.kr`

DEPARTMENT OF MATHEMATICAL SCIENCES, SEOUL NATIONAL UNIVERSITY, SEOUL, 151-747,
KOREA

E-mail address: `kkang@snu.ac.kr`

PARC, SEOUL NATIONAL UNIVERSITY, SEOUL, 151-747, KOREA

E-mail address: `hoya4200@snu.ac.kr`